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# Remarks on algebraic convergence of discrete Möbius groups

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## 1 Theorems of algebraic convergence of discrete groups

In 1982, T. Jørgensen and P. Klein proved the following result on algebraic convergence of a sequence of non-elementary finitely generated Kleinian groups.

**THEOREM 1.** ( Jørgensen-Klein [3] ) *Let  $\{G_m\}$  be a sequence of non-elementary  $r$ -generator Kleinian groups converging algebraically to the group  $G$ . Then  $G$  is also a non-elementary Kleinian group and the correspondence from the generators of  $G$  to their approximants in  $G_m$  extends for all sufficiently large  $m \in \mathbb{N}$  to a homomorphism of  $G$  onto  $G_m$ .*

Theorem 1 is an extension of the preceding theorem of the first author ( [2] ) in 1976. Main tool to establish these two theorems is the following proposition which is known as Jørgensen's inequality.

**PROPOSITION 2.** ( Jørgensen's inequality [2] ) *Let  $f$  and  $g$  be two linear fractional transformations which generate a non-elementary discrete group. Then the following inequality holds*

$$| \operatorname{tr}[f, g] - 2 | + | \operatorname{tr}^2(f) - 4 | \geq 1.$$

Attempts to extend Jørgensen's inequality to all dimensions were made in several manners. ( For example see [1] and [4]. ) In 1989, G.J. Martin showed a theorem on algebraic convergence of a sequence of non-elementary finitely generated discrete Möbius groups in several dimensions by use of his generalization of Jørgensen's inequality. In the case of several dimensions, the uniform bound of the order of elliptic cyclic groups in a sequence of Möbius groups plays an important role .

**THEOREM 3.** ( Martin [3] ) *Let  $G$  be the algebraic limit of a sequence  $\{G_m\}$  of non-elementary  $r$ -generator discrete subgroups of  $M(B^n)$  of uniformly bounded torsion. Then  $G$  is a non-elementary discrete group.*

In this note, we clarify the difference between two convergence theorems ( Theorem 1 and Theorem 2 ) by constructing some examples.

## 2 Examples

We need some notations and definitions. The unit ball  $B^n$  ( $n = 2, 3, 4, \dots$ ) in  $R^n$  with the Poincaré metric is a model of the  $n$ -dimensional hyperbolic space. Let  $M(B^n)$  be a subgroup of the general Möbius group  $M(\bar{R}^n)$  which keeps  $B^n$  invariant. For  $f, g \in M(B^n)$  we set

$$D(f, g) = \sup\{|f(x) - g(x)| \mid x \in S^{n-1} = \partial B^n\}$$

and regard  $M(B^n)$  as a metric space. We say that a subgroup  $G$  of  $M(B^n)$  is a non-elementary group if  $G$  contains two elements of infinite order with distinct fixed points.

Let  $\{G_m\}$  be a sequence of subgroups of  $M(B^n)$  each with same finite number of generators  $\{g_{m,1}, g_{m,2}, \dots, g_{m,r}\}$  for  $m = 1, 2, \dots$ . If we have  $D(g_{m,i}, g_i) \rightarrow 0$  as  $m \rightarrow \infty$  and  $g_i \in M(B^n)$  for  $i = 1, 2, \dots$ , then we say that the sequence of groups  $\{G_m\}$  converges algebraically to the limit group  $G = \langle g_1, g_2, \dots, g_r \rangle$ . For any Möbius transformation  $g$ , we denote the order of  $g$  by  $\text{ord}(g)$ . Let  $\{G_i\}_{i \in I}$  be a family of groups. We say that  $\{G_i\}_{i \in I}$  has uniformly bounded torsion if there is an integer  $m_0$  with the following properties : if  $g \in G_i$  for some  $i$ , then  $\text{ord}(g) = \infty$  or  $\text{ord}(g) \leq m_0$ . It is important to note the order of elliptic elements of a sequence of subgroups of  $M(B^n)$ .

**EXAMPLE 1.** For  $n \geq 4$  we construct a sequence  $\{G_m\}$  of non-elementary discrete subgroups of  $M(B^n)$  which converges algebraically to a non-discrete subgroup. With no loss of generality, we may assume  $n = 4$ . Let  $G_0 = \langle g_1, g_2, \dots, g_r \rangle \subset M(B^2)$  be a purely hyperbolic non-elementary Fuchsian group and representing a compact Riemann surface, that is a surface group. The group  $G_0$  acts on  $B^2$  which is embedded in  $B^4$  by the map  $(x, y) \mapsto (x, y, 0, 0)$ . The action of each  $g \in G_0$  extends uniquely to  $B^4$  by requiring that the extension is hyperbolic. In this way  $G_0$  becomes a non-elementary finitely generated discrete subgroup of  $M(B^4)$ . Let

$$h_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_m & -\sin \theta_m \\ 0 & 0 & \sin \theta_m & \cos \theta_m \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix},$$

where  $2\pi/\theta_m$  is rational ( $m = 1, 2, \dots$ ),  $2\pi/\theta$  is irrational and  $\theta_m \rightarrow \theta$  as  $m \rightarrow \infty$ . We set

$G_m = \langle G_0, h_m \rangle$  and  $G = \langle G_0, h \rangle$ . Since every hyperbolic element has no rotation part and  $h_m (m = 1, 2, \dots)$  fixes every point of  $B^2 \hookrightarrow B^4$ ,  $h_m$  commutes to each  $g \in G_0$ . We can easily see that  $G_m (m = 1, 2, \dots)$  and  $G$  are non-elementary groups. Since  $G$  contains an elliptic element  $h$  of infinite order,  $G$  is not discrete.

Now we show that  $G_m (m = 1, 2, \dots)$  is discrete. It is well known that the following three statements are equivalent to each other : (i)  $G_m$  is a discrete group. (ii)  $G_m$  acts discontinuously on  $B^4$ . (iii)  $G_m$  is discontinuous at some point of  $B^4$ . So it suffices to show that  $G_m$  is discontinuous at the origin. Let  $B$  be an open ball centered at the origin whose radius is sufficiently small. Denote by  $b = B \cap B^2$ . Since  $G_m|_{B^2} = G_0$  acts discontinuously on  $B^2$  as a surface group,  $\{g \in G_0 \mid g(b) \cap b \neq \emptyset\}$  is trivial. Recall that  $h_m (m = 1, 2, \dots)$  commutes to any  $g \in G_0$ . So any element  $g \in G_m$  is written in the form  $g = \tilde{g} \circ (h_m)^k$  for some  $\tilde{g} \in G_0$  and  $k \in \mathbb{Z}$ . Let  $g_0$  be an element of  $G_m$  such that  $g_0(B) \cap B \neq \emptyset$ . Then  $g_0(b) \cap b \neq \emptyset$  and we obtain  $g_0 = (h_m)^j$  for some  $j \in \mathbb{Z}$ . Since  $h_m$  is elliptic of finite order, we conclude the subgroup  $\{g \in G_m \mid g(B) \cap B \neq \emptyset\}$  of  $G_m$  is finite for  $m = 1, 2, \dots$ . Therefore  $G_m$  is discontinuous at the origin. Here we obtain that  $\{G_m\}$  is a sequence of non-elementary finitely generated discrete groups converging algebraically to a non-elementary non-discrete group  $G$ . Since  $\text{ord}(h_m) \rightarrow \infty$  as  $m \rightarrow \infty$ , the sequence  $\{G_m\}$  has not uniformly bounded torsion. So Theorem 1 cannot be extended directly to several dimensional case.

Now we consider the three dimensional case. Let  $G_0, \theta_m, \theta$  be same as those in the four dimensional case. We embed  $B^2$  in  $B^3$  by the map  $(x, y) \mapsto (x, y, 0)$ . We set

$$h_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_m & -\sin \theta_m \\ 0 & \sin \theta_m & \cos \theta_m \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

and  $G_m = \langle G_0, h_m \rangle (m = 1, 2, \dots)$ ,  $G = \langle G_0, h \rangle$ . For any  $m$  there exists a hyperbolic element  $g \in G_m$ , so that  $g$  and  $h_m g h_m^{-1}$  have distinct fixed points. So  $G_m$  is non-elementary for  $m = 1, 2, \dots$ . We can easily see that for arbitrary small  $\varepsilon > 0$  there exist an integer  $m_0$  and  $\tilde{f}_m \in \langle h_m \rangle$  such that  $D(\tilde{f}_m, Id) < \varepsilon$  for every  $m \geq m_0$ . So we can deduce that there exist  $\tilde{f}_m \in \langle h_m \rangle$  and a hyperbolic element  $g_m \in G_m$  so that

$$| \text{tr}[\tilde{f}_m, g_m] - 2 | + | \text{tr}^2(\tilde{f}_m) - 4 | < 1$$

for any sufficiently large integer  $m$ . Note that hyperbolic elements  $g_m, \tilde{f}_m g_m \tilde{f}_m^{-1}$  are contained in  $\langle \tilde{f}_m, g_m \rangle$  and have distinct fixed points. Hence  $\langle \tilde{f}_m, g_m \rangle$  is a non-elementary group. So Jørgensen's inequality yields that  $\langle \tilde{f}_m, g_m \rangle$  is non-discrete for any sufficiently large  $m$  and so is  $G_m$ .

Another point to the above example, we can arrange that the elliptic elements converges to the identity.

EXAMPLE 2. In the first place we consider the four dimensional ( several dimensional )

case. Let  $G_0$  be a surface group and

$$h_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\pi/m) & -\sin(\pi/m) \\ 0 & 0 & \sin(\pi/m) & \cos(\pi/m) \end{pmatrix},$$

and  $h = E_4$ , the four dimensional unit matrix. We set  $G_m = \langle G_0, h_m \rangle$  ( $m = 1, 2, \dots$ ) and  $G = \langle G_0, h \rangle = G_0$ . We can conclude that  $G_m$  ( $m = 1, 2, \dots$ ),  $G$  are non-elementary discrete groups and  $G_m$  converges algebraically to  $G$ . Obviously we can see that  $\{G_m\}$  has not uniformly bounded torsion. In this case however the correspondence from generators of  $G$  to  $G_m$  cannot be extended to a homomorphism of  $G$  onto  $G_m$  for any  $m$ .

In the case  $n = 3$ , a sequence of non-elementary groups  $\{G_m\}$  converges to a non-elementary discrete group  $G$ . But for any sufficiently large  $m$ ,  $G_m$  is not discrete.

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